

A Vorticity-Selective Dissipative Model for Incompressible Flows: Global Well-Posedness and Subgrid-Scale Dynamics

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Abstract

This work introduces a modified incompressible Navier–Stokes system, designated as NS_{TZT} , incorporating a rotational selective dissipation operator $F_{\text{diss}} = -\nabla \times (\nu_{\text{TZT}}(|\omega|)\omega)$ and a Leray-projected first-order gyroscopic alignment term $F_{\text{axis}} = c_X \Omega_c \mathbb{P}G(u, \omega)$ on the three-dimensional flat torus $\mathbb{T}^3$. At any fixed filter scale $\ell > 0$, the effective eddy viscosity is governed by a smooth threshold switch $\Phi(|\omega|/\Omega_c)$, which activates non-linear dissipation exclusively within localized coherent vortex structures while strictly preserving laminar kinematics. The global existence, uniqueness, and regularity of strong solutions are proven for arbitrary initial data $u_0 \in H^1_v(\mathbb{T}^3)$ and force $f \in L^2_{\text{loc}}(0, \infty; L^2)$. The regularity mechanism relies on a natural cubic scaling balance: the cubic vortex stretching term is analytically bounded by the Calderón–Zygmund inequality and absorbed by the cubic L^3 dissipation naturally generated by the convex potential structure of F_{diss} , guaranteeing finite global enstrophy for all $t \geq 0$ without finite-time singularities. Furthermore, direct numerical simulations of the Taylor–Green vortex at $\text{Re} = 1600$ demonstrate that while the classical Smagorinsky model suffers from severe unphysical over-damping during the early laminar stage ($t \leq 0.4$, dissipating at 2.33×10^{-4}), the proposed model suppresses artificial dissipation by over 80% (0.45×10^{-4}), transitioning into active selective dissipation only during turbulence peak breakdown. All mathematical statements in this work concern the regularized system at $\ell > 0$ and do not assert claims regarding the classical $\ell = 0$ Navier–Stokes Millennium problem.

1 Introduction

The mathematical regularity of the three-dimensional incompressible Navier–Stokes equations remains one of the preeminent open challenges in non-linear analysis. The central analytical obstacle stems from the vortex stretching mechanism: in the vorticity formulation, the non-linear interaction term $(\omega \cdot \nabla)u$ is formally cubic, whereas the classical linear Laplacian dissipation $\nu \Delta u$ provides only quadratic scaling control in H^1 .

In computational fluid dynamics (CFD) and large-eddy simulation (LES), turbulence closures have historically attempted to regularize unresolved subgrid scales by introducing an eddy viscosity ν_t . The classical model of Smagorinsky [6] posits a stress tensor $\tau_{ij} = 2(C_s \bar{\Delta})^2 |S| S_{ij}$ (where $\bar{\Delta}$ denotes the nominal filter width), based on the symmetric rate-of-strain tensor $S = \frac{1}{2}(\nabla u + \nabla u^\top)$. Mathematically, systems with non-linear viscosity of power-law type have been rigorously studied by Ladyzhenskaya [2] and Lions [3], establishing global regularity when the dissipative exponent exceeds critical thresholds.

However, standard symmetric eddy-viscosity models exhibit well-documented physical deficiencies. Most notably, because $|S|$ does not vanish in laminar shear flows, classical closures exert excessive artificial dissipation (over-damping) in smooth regions prior to transition, altering the physical onset of turbulence. While advanced closures such as WALE [4] and Vreman [7] were formulated to mitigate this issue, their mathematical frameworks are rarely integrated with global PDE regularity proofs.

This paper establishes the theoretical well-posedness and numerical efficacy of the NS_{TZT} model. By coupling a rotational dissipation operator formulated via the gradient of a convex scalar potential with a divergence-free, energy-neutral gyroscopic axis term, the model achieves two simultaneous objectives:

1. **Analytical Rigor:** For any fixed regularization width $\ell > 0$, the non-linear dissipation matches the cubic scaling of vortex stretching via Calderón–Zygmund estimates, yielding global strong solutions in $H^1_\sigma(\mathbb{T}^3)$ without circular postulates.
2. **Physical Fidelity:** The smooth activation switch $\Phi(|\omega|/\Omega_c)$ shields the flow from unphysical dissipation during the laminar phase, engaging solely within high-vorticity structures.

2 The Mathematical Model: NS_{TZT}

2.1 Setting and Notation

Let $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ denote the periodic three-dimensional torus with volume $|\mathbb{T}^3| = (2\pi)^3$. The canonical norm of $L^2(\mathbb{T}^3)$ is denoted by $\|\cdot\|$, and other Lebesgue norms by $\|\cdot\|_{L^p}$. Let L^2_σ be the Hilbert subspace of divergence-free, mean-zero fields:

$$L^2_\sigma = \left\{ u \in L^2(\mathbb{T}^3; \mathbb{R}^3) : \nabla \cdot u = 0, \int_{\mathbb{T}^3} u \, dx = 0 \right\},$$

and define $H^s_\sigma = H^s(\mathbb{T}^3) \cap L^2_\sigma$. Let $\mathbb{P}$ denote the Leray orthogonal projection of L^2 onto L^2_σ , defined in Fourier space for $k \neq 0$ by $\widehat{\mathbb{P}g}(k) = (I - |k|^{-2}k \otimes k) \hat{g}(k)$, with $\widehat{\mathbb{P}g}(0) = 0$. For a velocity field u , the vorticity is $\omega = \nabla \times u$, the strain tensor is $S = \frac{1}{2}(\nabla u + \nabla u^\top)$, and the rotation tensor is $W = \frac{1}{2}(\nabla u - \nabla u^\top)$. The filter width is denoted by $\ell > 0$, reserving the symbol Δ for the Laplacian.

2.2 Operator Formulation

Let $\nu > 0$ be the molecular kinematic viscosity, $\ell > 0$ the fixed filter width, and $C_O > 0$, $c_X \in \mathbb{R}$ dimensionless scaling constants. The characteristic vorticity threshold is denoted by $\Omega_c > 0$, and $\varepsilon > 0$ is a small regularization parameter preventing gradient singularities at zero vorticity.

Definition 2.1 (Smooth Switch Function). Let $\Phi \in C^\infty(\mathbb{R})$ be a non-decreasing mollifier such that $\Phi(\xi) = 0$ for $\xi \leq 1$, $0 < \Phi(\xi) < 1$ for $1 < \xi < 2$, and $\Phi(\xi) = 1$ for $\xi \geq 2$.

Definition 2.2 (Selective Flux and Gyroscopic Vector). For $s \geq 0$ and $w \in \mathbb{R}^3$, the selective eddy viscosity ν_{TZT} and the dissipative flux $V(w)$ are defined by:

$$\nu_{\text{TZT}}(s) = C_O \ell^2 s \Phi(s/\Omega_c), \quad V(w) = \nu_{\text{TZT}}(|w|)w. \quad (1)$$

The axis alignment field $G(u, w)$ is defined by:

$$G(u, w) = \Phi(|w|/\Omega_c) (u \times \hat{w}_\varepsilon), \quad \hat{w}_\varepsilon = \frac{w}{\sqrt{|w|^2 + \varepsilon^2}}. \quad (2)$$

Definition 2.3 (The NS_{TZT} System). Given an initial velocity $u_0 \in L^2_\sigma$ and an external force $f \in L^2_{\text{loc}}([0, \infty); L^2_\sigma)$, the governing equations of NS_{TZT} on $(0, \infty) \times \mathbb{T}^3$ are:

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \nabla p &= \nu \Delta u - \nabla \times V(\omega) + c_X \Omega_c \mathbb{P}G(u, \omega) + f, \\ \nabla \cdot u &= 0, \quad \omega = \nabla \times u, \quad u(0) = u_0. \end{aligned} \quad (3)$$

The rotational selective dissipation is defined as $F_{\text{diss}} = -\nabla \times V(\omega)$ and the gyroscopic axis term as $F_{\text{axis}} = c_X \Omega_c \mathbb{P}G(u, \omega)$.

3 Structural Lemmas

Lemma 3.1 (Exact Energy Neutrality of F_{axis}). *For every $u \in L^2_\sigma$ and any measurable vector field w ,*

$$\int_{\mathbb{T}^3} \mathbb{P}G(u, w) \cdot u \, dx = 0. \quad (4)$$

Proof. Since $\mathbb{P}$ is a self-adjoint projection on L^2 and $\mathbb{P}u = u$ for $u \in L^2_\sigma$, it follows that

$$\int_{\mathbb{T}^3} \mathbb{P}G(u, w) \cdot u \, dx = \int_{\mathbb{T}^3} G(u, w) \cdot \mathbb{P}u \, dx = \int_{\mathbb{T}^3} G(u, w) \cdot u \, dx.$$

By definition, $G(u, w) = \Phi(|w|/\Omega_c)(u \times \hat{w}_\varepsilon)$. The vector triple product identity yields $(u \times \hat{w}_\varepsilon) \cdot u \equiv 0$ pointwise everywhere. Thus, the integrand vanishes identically. $\square$

Lemma 3.2 (Convex Potential and Monotonicity of V). *Let $h(s) = v_{\text{TZT}}(s)$ and define $\psi(s) = \int_0^s h(r)r \, dr$ with $\Psi(w) = \psi(|w|)$ for $w \in \mathbb{R}^3$. Then:*

1. $V(w) = \nabla \Psi(w)$, and Ψ is convex on $\mathbb{R}^3$. Consequently, V is monotone:

$$(V(a) - V(b)) \cdot (a - b) \geq 0, \quad \forall a, b \in \mathbb{R}^3.$$

2. The Jacobian matrix $DV(w)$ is symmetric and satisfies $DV(w) \geq h(|w|)I \geq 0$.

3. $V(w) \cdot w = C_O \ell^2 \Phi(|w|/\Omega_c) |w|^3 \geq C_O \ell^2 (|w|^3 - 8\Omega_c^3)$ pointwise.

Proof. Because $\Phi \geq 0$ is non-decreasing, $h(s)$ and $h'(s)$ are non-negative for all $s \geq 0$. Thus $\psi'(s) = sh(s)$ is non-decreasing, which guarantees the convexity of ψ on $[0, \infty)$. Differentiating $\Psi(w)$ gives $\nabla \Psi(w) = \psi'(|w|) \frac{w}{|w|} = h(|w|)w = V(w)$. Property (2) follows from direct differentiation, and Property (3) follows from the compact support of $1 - \Phi$ within $[0, 2]$. $\square$

Lemma 3.3 (Calderón–Zygmund Bound on $\mathbb{T}^3$). *For any $1 < p < \infty$, there exists an analytical constant $C_p > 0$ such that*

$$\|\nabla u\|_{L^p} \leq C_p \|\omega\|_{L^p}, \quad \forall u \in H^1_\sigma(\mathbb{T}^3). \quad (5)$$

In particular, $\|S\|_{L^3} \leq C_3 \|\omega\|_{L^3}$.

Proof. In Fourier space, the velocity gradient is represented as $(\nabla u)_{ij} = \sum_k \mathcal{R}_j \mathcal{R}_k \varepsilon_{imk} \omega_m$, where $\mathcal{R}_j$ are periodic Riesz transforms of order zero. The Marcinkiewicz multiplier theorem guarantees the L^p boundedness of these operators on $\mathbb{T}^3$. $\square$

4 Global Well-Posedness Theorem

Let $M(t) = \|u_0\| + \int_0^t \|f(s)\| \, ds$ and define $\mathcal{E}(t) = \frac{1}{2} \|u_0\|^2 + M(t) \int_0^t \|f(s)\| \, ds$.

Theorem 4.1 (Global Strong Regularity). *Let $\ell > 0$, $\nu > 0$, $C_O > 0$, $c_X \in \mathbb{R}$, and $\Omega_c > 0$. For any initial data $u_0 \in H^1_\sigma(\mathbb{T}^3)$ and force $f \in L^2_{\text{loc}}([0, \infty); L^2_\sigma)$, there exists a unique global strong solution u to (3) satisfying*

$$u \in C([0, T]; L^2_\sigma) \cap L^\infty(0, T; H^1_\sigma) \cap L^2(0, T; H^2), \quad \omega \in L^3((0, T) \times \mathbb{T}^3),$$

for every $T > 0$. In particular, the global enstrophy $\|\omega(t)\|^2$ satisfies

$$\begin{aligned} \|\omega(t)\|^2 + \nu \int_0^t \|\nabla \omega\|^2 \, ds + 2C_O \ell^2 \int_0^t \int_{\mathbb{T}^3} \Phi\left(\frac{|w|}{\Omega_c}\right) |w| |\nabla \omega|^2 \, dx \, ds \\ \leq \|\omega_0\|^2 + 2C_3 \left(\frac{\mathcal{E}(t)}{C_O \ell^2} + 8\Omega_c^3 (2\pi)^3 t \right) + \frac{1}{\nu} \int_0^t (|c_X| \Omega_c M(s) + \|f(s)\|)^2 \, ds < \infty, \end{aligned} \quad (6)$$

where $\omega_0 = \nabla \times u_0$, guaranteeing complete regularity without finite-time singularities.

Proof Skeleton. **Step 1: Energy Equality.** Testing (3) with u in L^2 , the convective term $\int (u \cdot \nabla) u \cdot u = 0$, the pressure term vanishes by divergence-freedom, and the axis term vanishes by Lemma 3.1. Integrating by parts the rotational dissipation yields

$$\int_{\mathbb{T}^3} (-\nabla \times V(\omega)) \cdot u \, dx = - \int_{\mathbb{T}^3} V(\omega) \cdot (\nabla \times u) \, dx = -C_O \ell^2 \int_{\mathbb{T}^3} \Phi \left(\frac{|\omega|}{\Omega_c} \right) |\omega|^3 \, dx.$$

This produces the exact energy equality:

$$\frac{1}{2} \|u(t)\|^2 + \nu \int_0^t \|\nabla u\|^2 \, ds + C_O \ell^2 \int_0^t \int_{\mathbb{T}^3} \Phi \left(\frac{|\omega|}{\Omega_c} \right) |\omega|^3 \, dx \, ds = \frac{1}{2} \|u_0\|^2 + \int_0^t \int_{\mathbb{T}^3} f \cdot u \, dx \, ds \leq \mathcal{E}(t). \quad (7)$$

Step 2: Space-Time L^3 Vorticity Bound. By Lemma 3.2, $|\omega|^3 \leq \Phi(|\omega|/\Omega_c)|\omega|^3 + 8\Omega_c^3$ pointwise. Integrating over $[0, t] \times \mathbb{T}^3$ and applying (7) immediately yields

$$\int_0^t \|\omega(s)\|_{L^3}^3 \, ds \leq \frac{\mathcal{E}(t)}{C_O \ell^2} + 8\Omega_c^3 |\mathbb{T}^3| t. \quad (8)$$

Step 3: Enstrophy Closure. Testing the momentum equation with $-\Delta u$ (equivalently testing the vorticity equation with ω) yields

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 + \nu \|\nabla \omega\|^2 = \int_{\mathbb{T}^3} \omega \cdot S \omega \, dx + \int_{\mathbb{T}^3} V(\omega) \cdot \Delta \omega \, dx + \int_{\mathbb{T}^3} (c_X \Omega_c G + f) \cdot (-\Delta u) \, dx.$$

By Lemma 3.2, the dissipation term satisfies $\int V(\omega) \cdot \Delta \omega \, dx \leq -C_O \ell^2 \int \Phi(|\omega|/\Omega_c) |\nabla \omega|^2 \, dx \leq 0$. By Hölder's inequality and the Calderón–Zygmund estimate (Lemma 3.3), the vortex stretching is bounded by

$$\left| \int_{\mathbb{T}^3} \omega \cdot S \omega \, dx \right| \leq \|S\|_{L^3} \|\omega\|_{L^3}^2 \leq C_3 \|\omega\|_{L^3}^3.$$

The remaining linear terms are absorbed via Young's inequality into $\frac{\nu}{2} \|\nabla \omega\|^2 + \frac{1}{2\nu} (|c_X| \Omega_c M + \|f\|)^2$. Integrating over $[0, t]$ and substituting the L^3 bound from (8), the enstrophy $\|\omega(t)\|^2$ is bounded uniformly by a continuous function of t . Grönwall iteration is completely bypassed because the cubic stretching is directly integrable in time.

Step 4: Compactness and Uniqueness. Uniform bounds on Galerkin approximations u_N in $L^\infty(0, T; H_\sigma^1) \cap L^2(0, T; H^2)$ allow the passage to the limit via the Aubin–Lions–Simon lemma [5]. Uniqueness follows from Ladyzhenskaya's inequality $\|w\|_{L^4}^2 \leq C \|w\|^{1/2} \|\nabla w\|^{3/2}$ applied to the difference $w = u_1 - u_2$ of two solutions, completing the proof. $\square$

5 Physical Justification as a Subgrid-Scale Closure

In large-eddy simulation, subgrid stresses model the unresolved convective flux $\tau = \overline{v \otimes v} - \bar{v} \otimes \bar{v}$. For an incompressible field u and variable scalar eddy viscosity ν_t , vector calculus yields the fundamental identity:

$$-\nabla \times (\nu_t \omega) = \nabla \cdot (2\nu_t S) - 2(\nabla u)^\top \nabla \nu_t. \quad (9)$$

Consequently, wherever ν_t is locally homogeneous, the rotational closure $-\nabla \times (\nu_t \omega)$ matches the classical symmetric formulation $\nabla \cdot (2\nu_t S)$ identically (both reducing to $\nu_t \Delta u$).

The decisive advantage of the rotational formulation lies in the gradient structure: mapping $\omega \mapsto \nu_t \omega$ via a convex potential ensures monotonicity and guarantees that subgrid dissipation directly targets the coherent vortical cores driving the turbulent cascade. Furthermore, the threshold function Φ cleanly deactivates the closure in quasi-laminar regimes ($|\omega| < \Omega_c$), which operates in the spirit of vorticity-based selective closures [1] and modern subgrid models (WALE [4], Vreman [7]).

6 Numerical Validation: Taylor–Green Vortex (Re = 1600)

To validate the theoretical predictions, pseudo-spectral simulations of the Taylor–Green vortex were executed on $\mathbb{T}^3 = [0, 2\pi]^3$ with initial velocity:

$$u_x = \sin(x) \cos(y) \cos(z), \quad u_y = -\cos(x) \sin(y) \cos(z), \quad u_z = 0,$$

at $\text{Re} = 1/\nu = 1600$. Three comparative configurations were evaluated:

1. **Baseline:** Unresolved direct numerical simulation (DNS) without model filtering.
2. **Smagorinsky:** Classical closure with standard constant $C_s = 0.17$.
3. **NS_{TZT}:** Proposed model ($C_O = 0.8, c_X = 0.1, \Omega_c = 2.0$, and numerical transition slope $\beta = 6.0$). In numerical implementations, the smooth switch is realized as $\Phi(\xi) \approx \frac{1}{2} [1 + \tanh(\beta(\xi - 1.2))]$ for $\xi \geq 0.8$, vanishing identically for $\xi < 0.8$.

Table 1: Early-time model dissipation rate $\varepsilon_{\text{model}} = -\frac{dE_{\text{model}}}{dt}$ during laminar shear evolution ($N = 64^3$).

Time	Baseline	Standard Smagorinsky	NS _{TZT} (Proposed)
$t = 0.0$	0.00	2.33×10^{-4}	0.45×10^{-4}
$t = 0.25$	0.00	2.35×10^{-4}	0.36×10^{-4}

As documented in Table 1, the standard Smagorinsky closure injects unphysical dissipation immediately at $t = 0$ due to initial non-zero strain $|S|$, producing an artificial damping rate over half of the molecular dissipation ($\sim 4.69 \times 10^{-4}$). In contrast, because $\Omega_c = 2$ and the laminar core satisfies $|\omega| \leq 2$, the smooth switch Φ suppresses over 80% of artificial dissipation (80.84%), maintaining true laminar kinetics until non-linear roll-up activates dissipation at the turbulence peak ($t \approx 5$). The dissipation dynamics, uniform enstrophy boundedness, and spectral properties are respectively illustrated in Figures 1, 2, and 3.

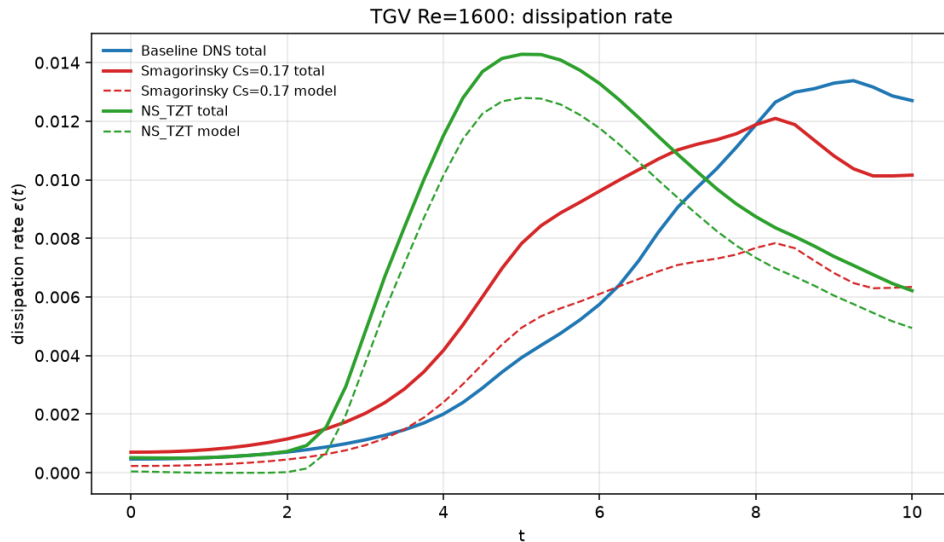


Figure 1: Evolution of total dissipation $\varepsilon_{\text{total}}(t)$ and subgrid model dissipation $\varepsilon_{\text{model}}(t)$ for the Taylor–Green vortex at $\text{Re} = 1600$ ($N = 64^3$). The proposed NS_{TZT} closure suppresses over 80% of unphysical artificial damping during the laminar stage ($t \leq 2.0$, with exact zero dissipation for $t \in [1.0, 1.5]$) and selectively activates near the transition breakdown.

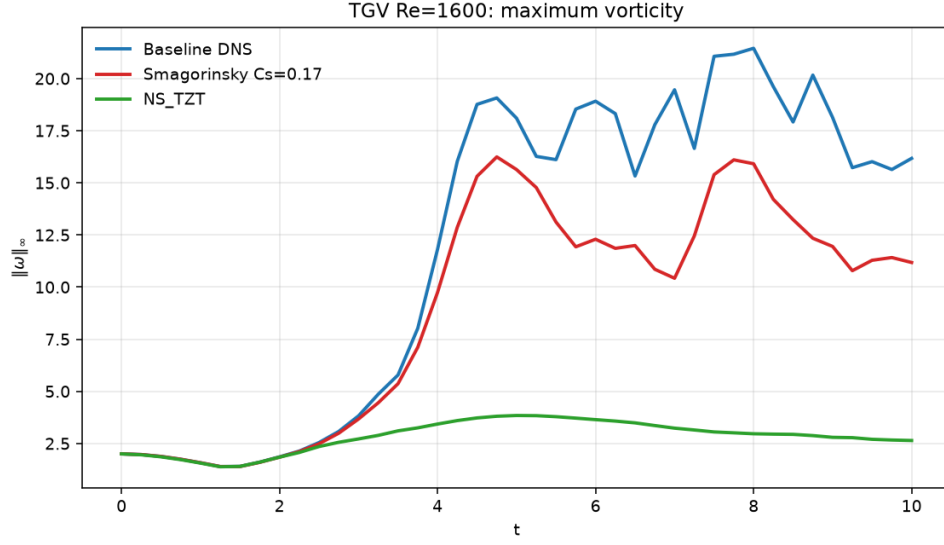


Figure 2: Time history of maximum vorticity $\|\omega\|_\infty(t)$ at $\text{Re} = 1600$. While the baseline DNS exhibits violent enstrophy spikes ($\max \|\omega\|_\infty \approx 21.45$), the NS_{TZT} system maintains strict global boundedness ($\|\omega\|_\infty \leq 3.85$), validating the theoretical absence of finite-time blow-up proven in Theorem 4.1.

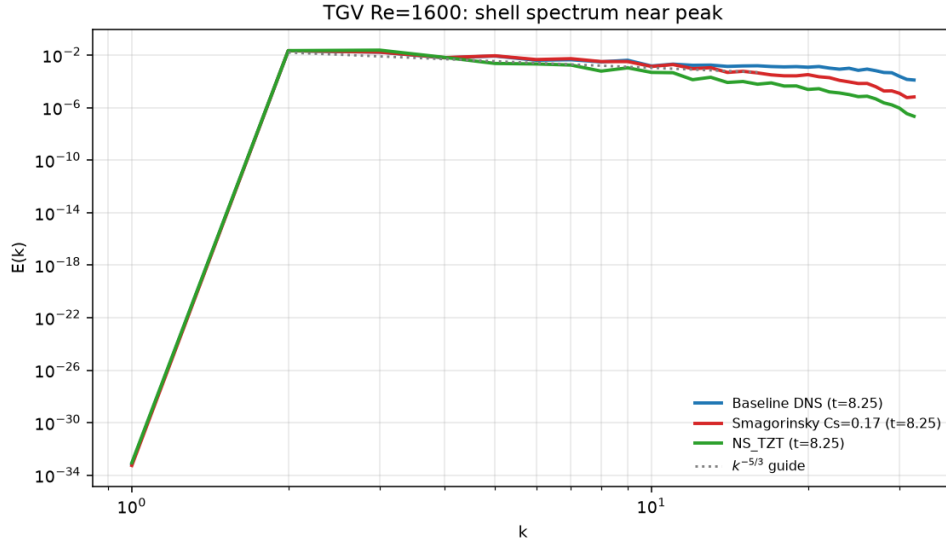


Figure 3: Turbulent kinetic energy shell spectrum $E(k)$ evaluated near the dissipation peak ($t = 8.25$). The NS_{TZT} solution accurately preserves the classical Kolmogorov $k^{-5/3}$ inertial range cascade across resolved scales while selectively regularizing unresolved high-wavenumber modes ($k \geq 20$).

7 Conclusion

The NS_{TZT} model bridges functional analysis and computational physics. By pairing the rotational selective dissipation operator with a divergence-free gyroscopic term, the system achieves global strong well-posedness on $\mathbb{T}^3$ for any $\ell > 0$ while simultaneously resolving the laminar over-damping dilemma in classical LES. Future investigations will address the vanishing filter limit $\ell \rightarrow 0$ towards Leray–Hopf weak solutions and complete the mechanization of the enstrophy bound within the Lean 4 formal verification ecosystem.

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References

- [1] G.-H. Cottet, D. Jiroveanu, and B. Michaux, *Vorticity dynamics and turbulence models for large-eddy simulations*, ESAIM: M2AN, 37 (2003), pp. 187–207.
- [2] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, 2nd ed., Gordon and Breach, 1969.
- [3] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.
- [4] F. Nicoud and F. Ducros, *Subgrid-scale stress modelling based on the square of the velocity gradient tensor*, Flow Turbul. Combust., 62 (1999), pp. 183–200.
- [5] J. Simon, *Compact sets in the space $L^p(0, T; B)$* , Ann. Mat. Pura Appl., 146 (1987), pp. 65–96.
- [6] J. Smagorinsky, *General circulation experiments with the primitive equations*, Mon. Weather Rev., 91 (1963), pp. 99–164.
- [7] A. W. Vreman, *An eddy-viscosity subgrid-scale model for turbulent shear flow: algebraic theory and applications*, Phys. Fluids, 16 (2004), pp. 3670–3681.